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The classification of log del Pezzo surfaces and their quasi-universal coverings

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We Work over an algebraically closed field k of characteristic zero. Let V be a nonsingular projective surface over k and D a reduced effective divisor on V with only simple normal crossings.

Write $D = \sum_{i=1}^{n} D_i$, where D_i's are irreducible components.

Definition 1 (cf. [6]). $Bk(D)=\sum_{i=1}^{n} \alpha_i D_i$ *is defined as the effective* $\mathbb{Q}$*-divisor satisfying :*

(i) Supp $Bk(D)$ *consists of all connected components of* D *which are contractible to quotient singularities and which contains no* (-1) *curves, and all admissible rational twigs* T_i's *in* D, *i.e., by definition,* $T:=T_i$ *is written as* $T= G_1+\cdots+G_s$ *such that:* $G_i \cong \mathbb{P}^1$, $(G_i^2)\leq -2$, $(G_i, G_j)\leq 1$ *and* $(G_i, G_j)=1$ *if and only if* $j-i=\pm 1$, $D-T$ *does not meet* $G_1+\cdots+G_{s-1}$ *when* $s\geq 2$.

(ii) $(D-Bk(D)+K_V, D_i) = 0$, *for every* $D_i \subseteq$ Supp $Bk(D)$.

Then $0\leq\alpha_i\leq 1$, *where* $i=1,2,\cdots,n$.

Denote by $D^{\#}$ *the effective* $\mathbb{Q}$*-divisor* $D-Bk(D) = \sum_{i=1}^{n} \beta_i D_i$

with $\beta_i = 1-\alpha_i (1\leq i\leq n)$.

If $\bar{\kappa}(V-D)\geq 0$, $D+K_V = (D^{\#}+K_V)+Bk(D)$ is the Zariski decomposition, where $D^{\#}+K_V$ is the numerically effective part and $Bk(D)$ is the negative definite part.

Lemma 2 (cf. [6]). *We have* $0\leq\beta_i\leq 1$. *More precisely,*
(i) $\beta_i=0$ *if and only if every connected component* F *of* D *containing* D_i *is contractible to a rational double point and there are no* (-1) *curves in* F.
(ii) $\beta_i<1$ *if and only if* $D_i \subseteq \mathrm{Supp}\, Bk(D)$.

In [6] M. Miyanishi and S. Tsunoda constructed an almost minimal model $\sigma: (V,D) \to (\tilde{V},\tilde{D})$ for each pair (V,D). $(\tilde{V},\tilde{D})$ is almost minimal. Namely, for every irreducible curve $\tilde{E}$ on $\tilde{V}$, either $(\tilde{E},\tilde{D}^{\#}+K_{\tilde{V}})\geq 0$ or the intersection matrix of $\tilde{E}+Bk(\tilde{D})$ is not negative definite. $(\tilde{V},\tilde{D})$ has properties : (i) $\sigma_*(D) = \tilde{D}$, (ii) $\bar{\kappa}(\tilde{V}-\tilde{D}) = \bar{\kappa}(V-D)$, etc.

In virtue of this result,we shall assume in the following arguments that (V,D) is almost minimal.

Y. Kawamata proved (cf. [5]) :

Suppose (V,D) is almost minimal. Then $D^{\#}+K_V$ is numerically effective if and only if $\bar{\kappa}(V-D)\geq 0$.

By the study done by M.Miyanishi and S.Tsunoda(cf.[6,7]), we know well the structure of a pair (V,D) with $\bar{\kappa}(V-D)=-\infty$, unless (V,D) is the following log del Pezzo surface of rank one with contractible boundaries; in this case V is rational.

Definition 3. *A pair* (V,D) *is called* **a log del Pezzo surface with contractible boundaries** *if*

(i) D *is contractible to quotient singular points on a projective normal surface by the contraction morphism* $g: V \to \bar{V}$; *there are no* (-1) *curves in* D, *i.e.,* g *is a minimal resolution of* $\mathrm{Sing}(\bar{V})$;

(ii) *the anticanonical divisor* $-K_{\bar{V}}$ *is ample.*

Moreover, if the Picard number $\rho(\bar{V}) := \mathrm{rk}(\mathrm{Pic}(\bar{V}) \otimes_{\mathbb{Z}} \mathbb{Q}) = 1$, (V,D) *is said to be* **of rank one.**

We often confuse (V,D) *with* $\bar{V}$.

Let (V,D) and $g: V \to \bar{V}$ be the same as in Definition 3. Then $-(D^{\#}+K_V)$ $(\equiv g^*(-K_{\bar{V}}))$ is numerically effective as a $\mathbb{Q}$-divisor and for every curve E on V, $-(E,D^{\#}+K_V)=0$ if and only if $E \le D$. Fix a natural number P such that $PD^{\#}$ is integral. Then $-P(G,D^{\#}+K_V) \in \mathbb{Z}_+$ for every curve G. Thus, we can fix an irreducible curve C such that $-(C,D^{\#}+K_V)$ attains the smallest positive value.

Lemma 4 (M. Miyanishi). (I) *If* $|C+D+K_V| \ne \phi$ *(called the* **case(α)**)*, then* $(V,C+D)$ *is a quasi-Iitaka surface(see* Definition 5 *below) by replacing* C *by a member of* $|C|$ *if necessary.*
(II) *If* $|C+D+K_V| = \phi$ *and* $(V,D) \ne (\Sigma_n, M_n)$ *where* Σ_n *is a*

Hirzebruch surface of deg n *and* M_n *is its minimal section (called the* **case(β)**), *then we may assume that* C *is a* (-1) *curve by replacing* C *by a curve* C' *with* $|C'+D+K_V| = \phi$ *and* $-(C', D^{\#}+K_V) = -(C, D^{\#}+K_V)$.

Definition 5. *A pair* (W,B) *with* W *rational is said to be a* **quasi-Iitaka surface** *if there exists a decomposition of* B *into reduced effective integral divisors* $B = A+N$ *such that*

(i) $A+K_W \sim 0$,

(ii) N *is contractible to rational double singular points; there are no* (-1) *curves in* N, *hence* (Supp N) $\cap$ (Supp A) $= \phi$.

Moreover, if B *has only simple normal crossings,* (W,B) *is called* **an Iitaka surface.**

We sketch how Iitaka surfaces are classified (cf. [10]).

Let $h: W \to \bar{W}$ be the contraction of N to rational double singular points. Then, since $K_{\bar{W}} \sim -h_* A$, $K_{\bar{W}}$ is not numerically effective. Applying the Mori theory, we can find an extremal rational curve $\bar{\ell}$ on $\bar{W}$. Namely, $-3 \leq (\bar{\ell}, K_{\bar{W}}) < 0$, and if Z_1 and Z_2 are in the closure of the cone $\overline{NE}(\bar{W})$ of effective 1-cycles and $Z_1+Z_2 \in \mathbb{R}_+[\bar{\ell}]$, then $Z_1, Z_2 \in \mathbb{R}_+[\bar{\ell}]$. There exists a numerically effective divisor $\bar{H}$ on $\bar{W}$ such that $\bar{H}^{\perp} \cap \overline{NE}(\bar{W}) = \mathbb{R}_+[\bar{\ell}]$.

Case(1) $\bar{H} \equiv 0$. Then $\rho(\bar{W}) = 1$ and $-K_{\bar{W}}$ is ample.

Lemma 6. *If* $\rho(\bar{W})=1$ *(hence* $\rho(W) = \#(N)+1$*) then* $\bar{W}$ *is a rational Gorenstein log del Pezzo surface of rank 1. So,* N *has one of the following* 27 *Dynkin types(cf.* [4], [9], [3], [10]).

A_1, A_1+A_2, A_4, $2A_1+A_3$, D_5, A_1+A_5, $3A_2$, E_6, $3A_1+D_4$, A_7, A_1+D_6, E_7, A_1+2A_3, A_2+A_5, D_8, $2A_1+D_6$, E_8, A_1+E_7, A_1+A_7, $2A_4$, A_8, $A_1+A_2+A_5$, A_2+E_6, A_3+D_5, $4A_2$, $2A_1+2A_3$, $2D_4$.

Case(2) $\bar{H}\neq 0$ *and* $(\bar{H}^2)=0$. Then $\bar{H} \in \mathbb{R}_+[\bar{\ell}]$ and $(\bar{\ell}^2)=0$. Suppose $W \not\cong \mathbb{P}^2$ or Σ_n. Then, for $N \gg 0$ such that $N\bar{\ell}$ is a Cartier divisor, $\Phi_{|h*N\bar{\ell}|}$ is composed with a $\mathbb{P}^1$-fibration Φ: $W \to \mathbb{P}^1$. Every singualr fiber of Φ is written as $f_1 = 2(E+B_1+\cdots+B_{s-2})+B_{s-1}+B_s$, where E is a (-1) curve and B_i's are irreducible components of N.

Case(3) $(\bar{H}^2) > 0$. Then the proper transform $\ell=h'\bar{\ell}$ is a (-1) curve. Furthermore, if ℓ is not an irreducible component of A then the connected component R of $\ell+N$ containing ℓ is contractible to a smooth point. Indeed, R is a rod and ℓ is a tip. Let σ: $W \to \bar{W}$ be the contraction of ℓ if ℓ is an irreducible component of A and of R otherwise. Consider a quasi-Iitaka surface (W,σ_*B) and apply the Mori theory again. After a finitely many steps, we are reduced to the case (1) or (2) above.

This completes the classification of Iitaka surfaces. For quasi-Iitaka surfaces, this process works, too.

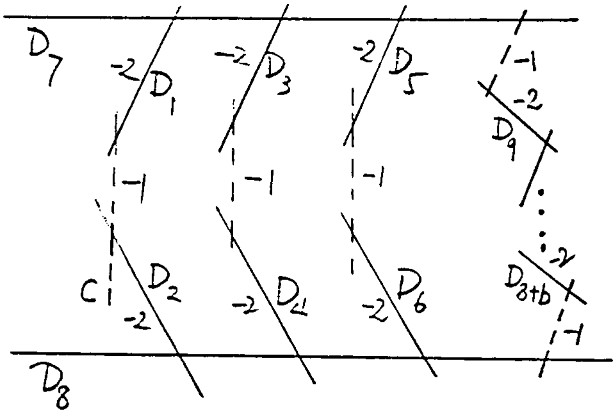
We now consider the case(β) of Lemma 4. We study this case according as how many irreducible components of D intersect with the (-1) curve C. We recall that a surface $V-D$ is called affine uniruled (resp. affine-ruled) if there exists a dominant morphism (resp. open immersion) $\mathbb{A}^1 \times G \to V-D$ with an affine curve G.

Theorem 7. *Suppose that the case(β) occurs, i.e.,* $|C+D+K_V| = \phi$ *and* $(V,D) \neq (\Sigma_n, M_n)$, *and that* $V-D$ *is not affine-ruled. Suppose the following case(β-1) does not occur.*

Case(β-1) The (-1) *curve* C *meets exactly two irreducible components* D_1 *and* D_2 *of* D *with* $(D_1^2)=-2$ *and* $(D_2^2) \leq -3$.

Then there exists a $\mathbb{P}^1$*-fibration* $\Phi: V \to \mathbb{P}^1$ *such that all singular fibers of* Φ *and the configuration of* $C+D$ *are precisely decribed as follows :*

Group(I). It consists of the following 8 Figures.



$$b = -(D_7^2) - (D_8^2) - 4$$

Figure (1)

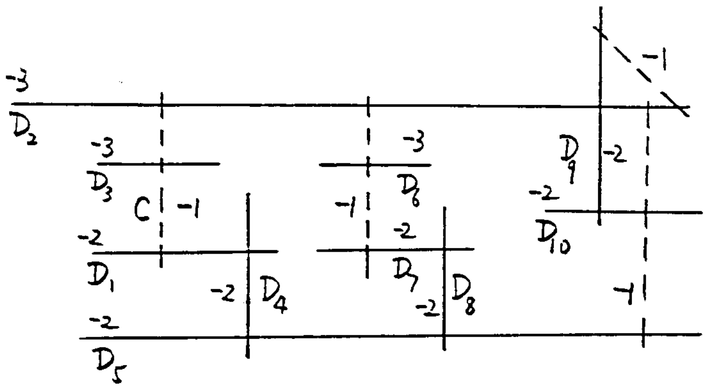


Figure (2)

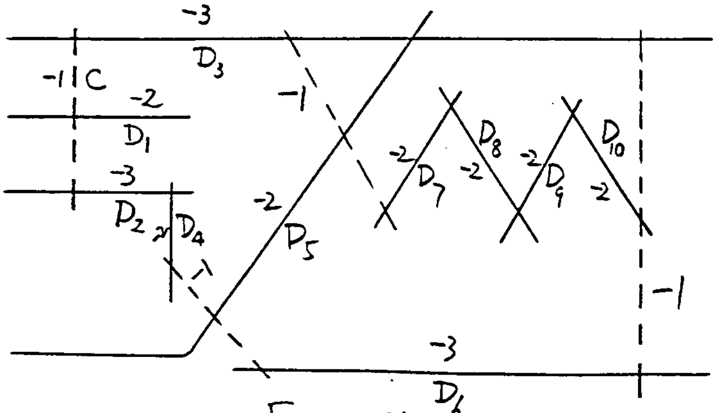


Figure (3)

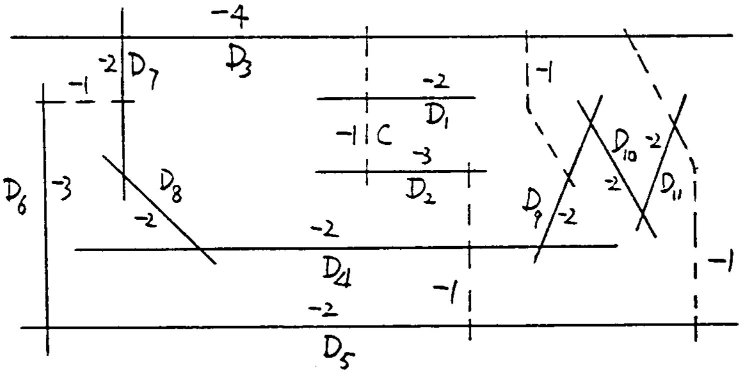


Figure (7)

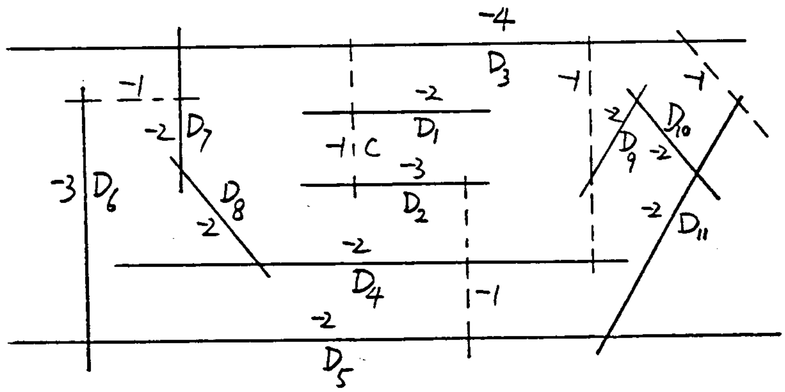
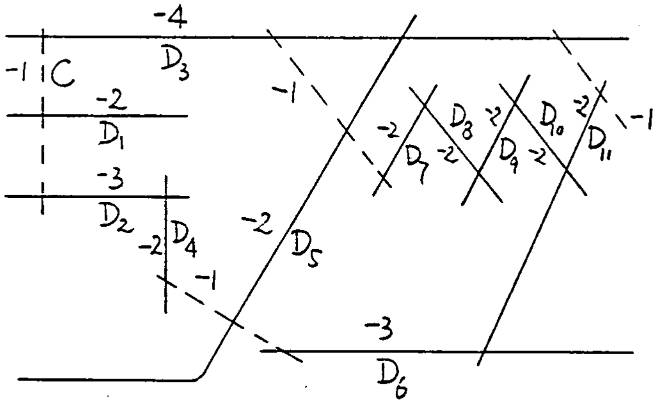


Figure (8)



Figure(4)

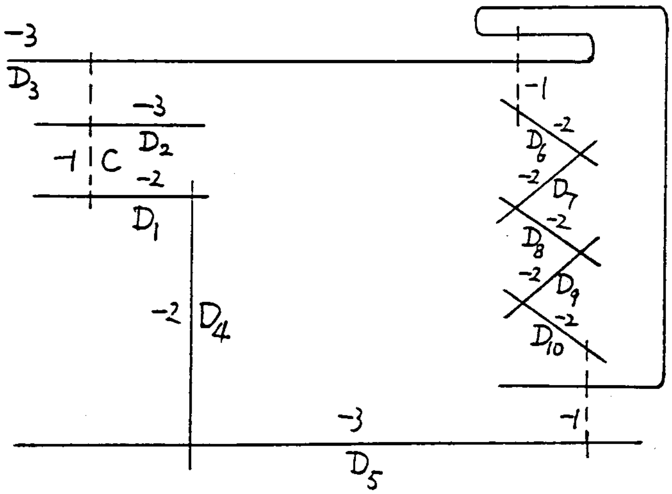


Figure (5)

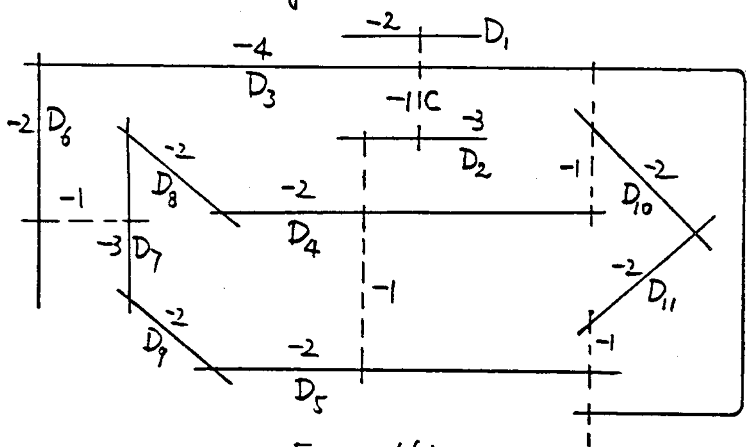
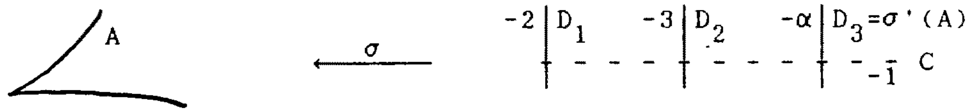


Figure (6)

Group(II). (V,D) is obtained by blowing up a quasi-Iitaka surface (W,B) in the way shown below, where B=A+N, A is a cuspidal curve in $|-K_W|$ and $\rho(W) = \#(N)+1$ (hence all possible Dynkin types of N are given in Lemma 6).



$(A^2) = 1$, 2 or 3; $\alpha = 6 - (A^2) = 5$, 4 or 3.

We set $D = D_1+D_2+D_3+\sigma^*(N)$.

Group(III). C meets only one component D_1 of D and the connected component Δ_1 of D containing D_1 is a fork. Moreover, either D_1 is the central component of Δ_1 with $(D_1^2) = -2$ or D_1 is contained in a twig T_1 of Δ_1 such that $C+T_1$ is negative definite. Let σ: V → W be the contraction of C in the first case and all contractible curves of $C+T_1$ in the second case. Then $(W,\sigma_*(D))$ is a log del Pezzo surface of rank one with *non*-contractible boundaries, M. Miyanishi and S. Tsunoda proved :

There exists a $\mathbb{P}^1$-fibration Ψ: W → $\mathbb{P}^1$ where all singular fibers and the configuration of $\sigma_*(D)$ are described precisely in [7; Lemma 2.6 and Theorems 4 and 6].

So, the configration of C+D and all singular fibers of $\Phi := \Psi\circ\sigma$: V → $\mathbb{P}^1$ can be obtained.

(*) The surfaces given in Figures 2 ~ 8 and Group(II) are deduced from our unpublished notes.

Lemma 8 (M.Miyanishi). *Suppose that the case(β-1) occurs, i.e.,* $|C+D+K_V| = \phi$, $(V,D) \neq (\Sigma_n, M_n)$ *and* C *meets exactly two irreducible components* D_1 *and* D_2 *of* D *with* $(D_1^2) = -2$ *and* $(D_2^2) \leq -3$. *Let* $\eta: V \to W$ *be the contraction of the* (-1) *curve* C. *Then* $(W, \eta_*(D-D_1))$ *is a log del Pezzo surface of rank one with contractible boundaries.*

Combining the above results, we get a method to classify log del Pezzo surfaces of rank one with contractible boundaries. Actually, in some easy cases, we can obtain all possibilities of pairs (V,D) fitting the case(β-1) by Lemma 8.

Definition 9. *A log del Pezzo surface* (V,D) *of rank one with contractible boundaries is called* **a dP3 surface** *if* $\bar{V}$, *with the contraction morphism* $g: V \to \bar{V}$, *has exactly one rational triple and several rational double singualr points.*

Let (V,D) be a dP3 surface fitting the case(β-1). Then $\eta_*(D-D_1)$ consists of (-2) curves and (-2) forks, where $\eta: V \to W$ is the contraction of C. By Lemma 2,(i), $-K_W = -((\eta_*(D-D_1))^{\#} + K_W)$, which is numerically effective. Note that $0 < (K_W^2) = 10 - \rho(W) = 9 - \#(\eta_*(D-D_1)) \leq 7$. So, W is obtained by blowing up $9-(K_W^2)(\leq 8)$ points on $\mathbb{P}^2$, which might include infinitely near points. Demazure proved that there is a nonsingular elliptic curve A in $|-K_W|$ (cf. [2; Theorem 1, p.39]). So, $(W, A+\eta_*(D-D_1))$ is an Iitaka surface with $\rho(W) = \#(\eta_*(D-D_1))+1$. Thus, $\eta_*(D-D_1)$

has one of 27 Dynkin types given in Lemma 6. By considering suitable $\mathbb{P}^1$-fibrations, we can work out all possibilities of pairs (V,D) fitting the case(β-1). At the same time we find a $\mathbb{P}^1$-fibration Ψ: $V \to \mathbb{P}^1$ for each pair (V,D), and write out explicitly the configuration of C+D and singular fibers of Ψ.

Now we let the field $k = \mathbb{C}$.

Theorem 10(*cf.*[12]). *Let* (V,D) *be a dP3 surface. Then we have:*

(I) *The configuration of* D *is No.n of Table 1* ($1 \le n \le 97$), *which is given at the end of the paper.*

(II) *We find a* $\mathbb{P}^1$*-fibration* Ψ: $V \to \mathbb{P}^1$ *and write out explicitly the configuration Picture(n) of* C+D *and all singular fibers of* Ψ(*cf.Picture(20) below*). *By reason of spaces, we omit them.*

(III) $\pi_1(V^0)$ *is a finite group. The quasi-universal covering* $\bar{U}$ *of* $\bar{V}$ (*cf.* Definition 11 *below*) *is a log del Pezzo surface with contractible boundaries and* $\bar{U}$ *is rational.*

(IV) *Suppose* $\pi_1(V^0)=(0)$. *Then* $V^0 \supseteq \mathbb{C} \times \mathbb{C}^*$ *where* $\mathbb{C}^* := \mathbb{C}-(0)$.

(V) *Suppose* $\pi_1(V^0) \neq (0)$. *Then* $\bar{V}$ *is the quotient of* $\mathbb{P}^2$ *by a finite subgroup* H *of* $PGL(2,\mathbb{C})$ *iff* $\rho(\bar{U}) = 1$. *If this is the case, there is a cyclic normal subgroup* H_1 *of* H *such that* $H/H_1 \cong \pi_1(V^0)$ *and* $\mathbb{P}^2/H_1 \cong \bar{U}$.

Definition 11. *Let* U^0 *be the topologically universal covering of* V-D. *If the fundamental group* $\pi_1(V-D)$ *is finite, then* U^0 *is algebraic and we let* $\bar{U}$ *be the normalization of*

$\bar{V}$ *in the function field* $k(U^0)$. $\bar{U}$ *is said to be the quasi universal covering of* $\bar{V}$, *which is unique by Zariski Main Theorem.*

In the joint work with M.Miyanishi we prove the following Theorem 12 (cf. [8; Theorems 1 and 2]).

Theorem 12. *Let* S *be a singular rational Gorenstein log del Pezzo surface with* $\rho(S)=1$. *Then the quasi-universal covering* $\tilde{S}$ *of* S *exists and* $\tilde{S}$ *is given in Table 2 at the end of the present paper.*
Moreover, we have :
(1) $S^0 := \mathrm{Reg}(S)$ *is simply connected,* i.e., $\pi_1(S^0)=0$ *if and only if* S *is a Gorenstein algebraic compactification of the affine plane* $\mathbb{C}^2$ *by an irreducible curve at infinity; for another proof of the "if" part, see* Brenton[1; Theorem 6].
(2) *Suppose* $\pi_1(S^0) \neq 0$. *Then* $S \cong \mathbb{P}^2/G$ *for a finite group* $G \subseteq PGL(2;k)$ *if and only if the Picard number* $\rho(\tilde{S}) = 1$.

Remark. Actually, we proved that S^0 is simply connected if and only if S is a Gorenstein algebraic compactification of the affine plane $\mathbb{C}^2$(the boundary might be *reducible*) under the assumption that S is a singular rational Gorenstein log del Pezzo surface (not necessarily $\rho(S) = 1$).

We want to construct the quasi-universal covering of a dP3 surface. We need the following Proposition 13 which is proved by the purity of the branch locus.

Proposition 13. *Let* Y *be a normal surface with only quotient singularities and let* $\pi: X \to Y$ *be a finite morphism which is étale outside* $\mathrm{Sing}(Y)$. *Then* X *has only quotient singularities. In particular, if* Y *is a log del Pezzo surface so is* X, *because* $-K_X = \pi^*(-K_Y)$.

Corollary 14. *Let* (V,D) *be a dP3 surface and* $g: V \to \bar{V}$ the *contraction of* D. *Suppose that there exist integral divisors* F *and* Δ *on* V *and an integer* $\ell \geq 2$ *such that* $\Delta > 0$, $\Delta \sim \ell F$ *and every irreducible components of* Δ *is contained in* D *and has coefficient less than* ℓ. *Denote by* $\tau: X \to V$ *the normalization of the covering surface which is defined by the relation* $\Delta \sim \ell F$ *and which hence branches exactly over* Δ. *Then* $\tau^{-1}(D)$ *is contractible to points on a projective normal surface* $\bar{X}$ *and* τ *induces a finite morphism* $\bar{\tau}: \bar{X} \to \bar{V}$ *such that* $\bar{\tau} = \tau$ *over* $\mathrm{Reg}(\bar{V})$ *(hence* $\bar{\tau}$ *is étale outside* $\mathrm{Sing}(\bar{V})$). *So,* $\bar{X}$ *is a log del Pezzo surface.*

For the computation of $H_1(V-D;\mathbb{Z})$ we use

Lemma 15 (cf.[8;Lemma 1]). *Let* (V,D) *be a dP3 surface. Then we have :*

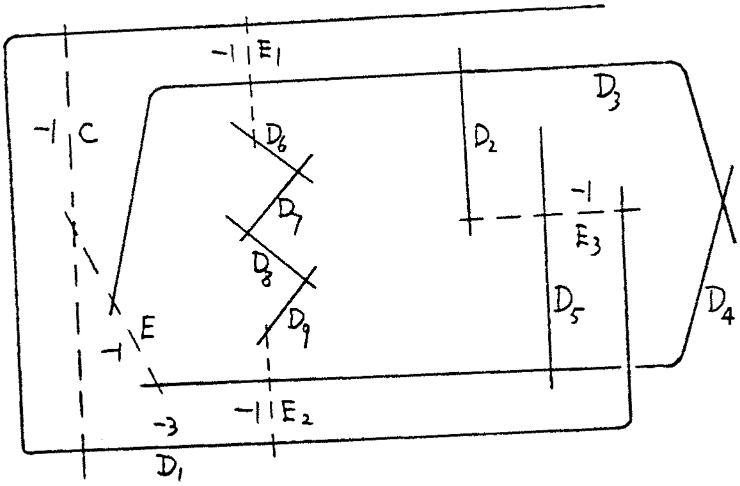
$$H_1(V-D;\mathbb{Z}) \cong (H^2(D;\mathbb{Z})/H_2(D;\mathbb{Z}))/(C\ell(\bar{V})/\mathrm{Pic}(\bar{V})).$$

We explain our method of finding the quasi-universal covering of a dP3 surface $\bar{V}$ by the following :

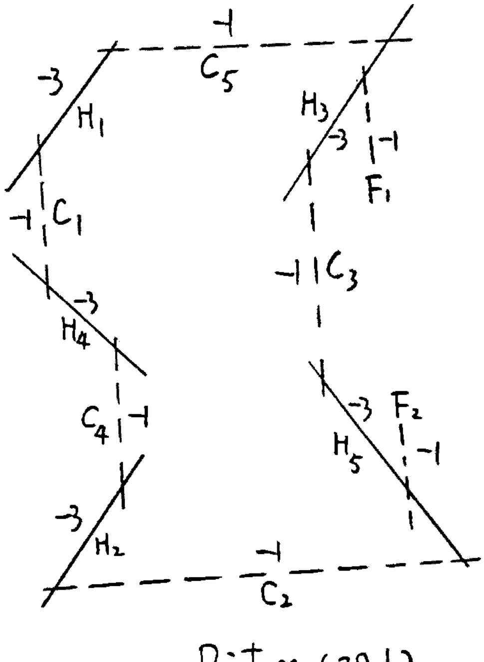
Example 16. Consider the No.20 surface (V,D) of Theorem 10 (cf.Picture(20) below). Let $v: V \to \Sigma_2$ be the contraction of curves in the singular fibers of the vertical fibration such that $(v_*D_3)^2 = -2$. Then we have :

$M:=v_*D_3$, $L:=v_*D_2$, $v_*D_4 \sim M+3L$, $v_*D_1 \sim 2M+4L$, $L \sim v_*E \sim v_*E_1$.

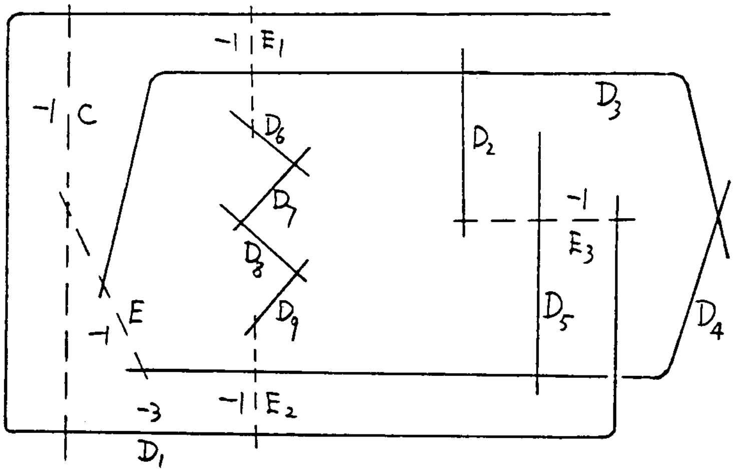
These implies $5v^*(M+L) \sim v^*v_*D_4+2v^*v_*D_2+4D_3$. Hence if we let $\Delta = 2D_2+4D_3+D_4+3D_5+D_6+2D_7+3D_8+4D_9$ and $F = v^*L+D_3-E_2-E_3$, then $\Delta \sim 5F$. By Corollary 14, there exists a finite morphism $\bar{\tau}: \bar{X} \to \bar{V}$ which is étale outside $\mathrm{Sing}(\bar{V})$ and which has $\deg \bar{\tau} = 5$ (where $g: V \to \bar{V}$ is the contraction of D). Let $h: U \to \bar{X}$ be the minimal resolution of $\mathrm{Sing}(\bar{X})$. Then $h^{-1}(\mathrm{Sing}\ \bar{X}) = \sum_{i=1}^{5} H_i$ (cf. Picture(20.1) below). We see $\bar{X} - \mathrm{Sing}(\bar{X}) = U - h^{-1}(\mathrm{Sing}\ \bar{X}) \supseteq \mathbb{C}^2$. Indeed, let $u: U \to \Sigma_0$ be the contraction of C_1, C_4, H_4, H_2, C_3, F_2, H_5 and H_3. Then $\Sigma_0 - u_*(\sum_{i=1}^{5} H_i) \supseteq \mathbb{C}^2$. So, $\bar{X} - \mathrm{Sing}(\bar{X})$ is simply connected. This implies that $\bar{\tau}^{-1}(\bar{V} - \mathrm{Sing}(\bar{V}))$ is simply connected. Hence $\bar{X}$ is the quasi-universal covering of $\bar{V}$ and $\pi_1(V-D) \cong \mathbb{Z}/5\mathbb{Z}$.



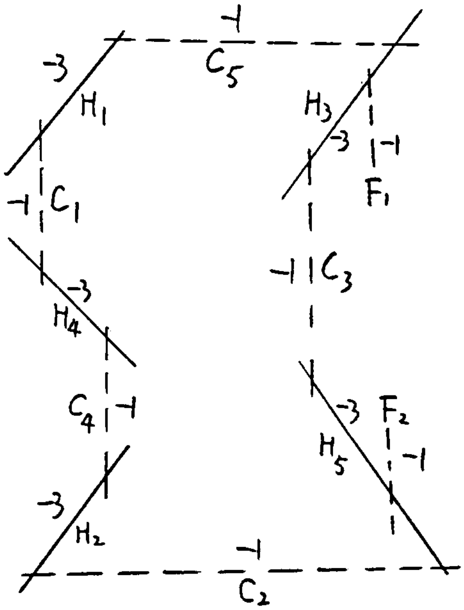
Picture (20)



Picture (20.1)



Picture (20)



Picture (20.1)

Table 1

No	Sing. type of $\bar{V}$	$H_1(V^0;\mathbb{Z})$	$\pi_1(V^0)$	$\rho(\bar{U})$	Sing. type of $\bar{U}$	Ruledness of V^0, U^0
1	$*$	(0)	(0)	1	$\bar{U} = \bar{V} = \bar{\Sigma}_3$	$V^0 \supseteq \mathbb{C}^2$
2	$A_4+(*-o^4)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
3	$*-o^8$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
4	$A_1+(*-o^7)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
5	$D_6+(o-*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_7+(-4)$	$U^0 \supseteq \mathbb{C}^2$
6	$2A_1+D_4+(o-*-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\lvert\pi_1\rvert=16$	2	$\bar{U} = \Sigma_0$	$U^0 = \Sigma_0$
7	$2A_3+(o-*-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	D_4 or Q_3	4	$2A_1$	$U^0 \supseteq \mathbb{C}^2$
8	$A_1+A_2+(*-o^5)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
9	$D_5+(*-o^3)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
10	$2A_1+A_3+(*-o^3)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	3	$3A_1+2*$	$U^0 \supseteq \mathbb{C}^2-P$
11	$A_1+A_5+(*-o^2)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_2+2(*-o-o)$	$U^0 \supseteq \mathbb{C}^2$
12	$E_6+(*-o^2)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
13	$A_1+D_6+(*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$D_4+2(*-o)$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
14	$A_7+(*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$A_3+2(*-o)$	$U^0 \supseteq \mathbb{C}^2$
15a	$E_7+(*-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
15b	$E_7+(*-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
16	D_8+*	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	D_5+2*	$U^0 \supseteq \mathbb{C}^2-P$
17	A_1+E_7+*	$\mathbb{Z}/2\mathbb{Z}$	S_3	4	D_4	$U^0 \supseteq \mathbb{C}^2$
18a	E_8+*	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
18b	E_8+*	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
19	A_1+A_7+*	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	5	A_1+4*	$U^0 \supseteq \mathbb{C}^2-P$
20	$2A_4+*$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	5	$5*$	$U^0 \supseteq \mathbb{C}^2$
21	A_8+*	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	5	A_2+3*	$U^0 \supseteq \mathbb{C}^2$

22	$A_1+A_2+A_5+*$	$\mathbb{Z}/6\mathbb{Z}$	$\lvert\pi_1\rvert=18$	4	smooth del Pezzo surface of deg 6	$U^0=\bar{U}$
23	$3A_2+(*-o^2)$	$\mathbb{Z}/3\mathbb{Z}$	$\lvert\pi_1\rvert=21$	1	$\bar{U}=\mathbb{P}^2$	$U^0=\mathbb{P}^2$
24	$A_2+A_5+(*-o)$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	3	$A_1+3(*-o)$	$U^0\supseteq\mathbb{C}^2$
25	A_2+E_6+*	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	3	D_4+3*	$U^0\supseteq\mathbb{C}^2$
26	A_3+D_5+*	$\mathbb{Z}/4\mathbb{Z}$	$\lvert\pi_1\rvert=12$	6	smooth del Pezzo surface of deg 4	$U^0=\bar{U}$
27	$A_1+2A_3+(*-o)$	$\lvert H_1\rvert=4$	$\lvert\pi_1\rvert=20$	2	$\bar{U}=\Sigma_0$	$U^0=\Sigma_0$
28	$2A_1+D_5+*$	$\mathbb{Z}/2\mathbb{Z}$	S_3	1	$\bar{U}=\bar{\Sigma}_2$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
29	$2A_1+(*-o^3-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	5	$*-o-*$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
30	$A_1+(*-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
31	$2A_1+(o-*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	1	$\bar{U}=\bar{\Sigma}_4$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
32	$A_1+(o-*-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$(-4)-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
33	$A_1+(o-*-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$(-4)-o-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
34	$3A_1+(o-*-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	1	$\bar{U}=\bar{\Sigma}_6$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
35	$2A_1+(o-o-o-*-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	3	$o-(-6)-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
36	$*-o-o-o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
37	$o-o-o-*-o-o-o$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$o-(-4)-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
38	$*-o-o-o-o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
39	$o-*-o-o-o$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
40	$A_1+(*-o^5)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
41	$A_1+(o-o-*-o^3)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
42	$*-o^7$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
43	$o-o-o-*-o^4$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
44	$A_2+(o-*-o^4)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
45	$o-*-o^7$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{*}$
46	$o-o-o-o-*-o^4$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supset\mathbb{C}\times\mathbb{C}^{*}$

47	$A_1+(o-o-*-o^5)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supset \mathbb{C}\times\mathbb{C}^*$
48	$o-o-o-*-o-o-o$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$o-(-4)-o-o$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
49	$o-o-o-*-o-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supset \mathbb{C}\times\mathbb{C}^*$
50	$*-o-o-o-o-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
51	$2A_1+(o-o-*-o^3)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	5	$*-(-4)-*$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^{**}$
52	$*-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
53	$*-o-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
54	$*-o-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
55	$o-*-o-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
56	$A_1+(*-o-o^4)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
57	$o-o-*-o-o$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
58	$A_1+(o-o-*-o^3)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$o-o-(-4)-o-o$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
59	$A_3+(o-o-o-*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_1+(o-o-o-*-o^3)$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
60	$D_4+(*-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_3+(*-o^3-*)$	$U^0 \supseteq \mathbb{C}^2$
61	$A_3+(*-o-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_1+(*-o^5-*)$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
62	$D_5+(*-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	6	$A_1+(-4)$	$U^0 \supseteq \mathbb{C}^2$
63	$A_2+(*-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
64	$A_1+A_2+(*-o-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
65	$A_2+(*-o-o-o-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
66	$A_3+ *$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}^2$
67	$A_1+A_4+ *$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
68	$A_6+ *$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$
69	$A_1+A_4+(*-o-o)$	(0)	(0)	1	$\bar{U} = \bar{V}$	$V^0 \supseteq \mathbb{C}\times\mathbb{C}^*$

70	$A_2+A_3+(*-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
71	$A_6+(*-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
72	$A_3+(*-o-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
73	$A_1+(o-*-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
74	$2A_1+(o-*-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$2A_1+((-4)-o)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
75	$A_1+(o-*-o^5)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$2A_1+((-4)-o-o)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
76	$A_1+A_2+(o^2-*-o^2)$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	3	$3A_1+(-5)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
77	$A_1+(o-o-*-o^5)$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	5	$3A_1+((-5)-o)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
78	$2A_1+(o^3-*-o^3)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	5	$4A_1+(-6)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
79	$A_2+A_3+(o-*-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$A_1+2A_2+(-4)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
80	$A_2+(o-*-o^5)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$2A_2+((-4)-o-o)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
81	$2A_2+(o-o-*-o-o)$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	3	$3A_2+(-5)$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
82	$A_1+A_3+(o-*-o^3)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$2A_3+(o-(-4))$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
83	$A_5+(o-*-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
84	A_3+D_4+*	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	A_1+A_3+2*	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
85	$A_1+(*-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
86	$A_1+(*-o-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}^2$
87	$2A_1+(*-o-o-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	$2A_1+((-3)-o-(-3))$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
88	$2A_1+A_2+(*-o-o)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	1	$\bar{U}=\bar{\Sigma}_4$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
89	$A_2+(*-o-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
90	$A_3+(*-o-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
91	$A_4+(*-o-o-o)$	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
92	$A_1+A_2+(*-o-o^3)$	$\mathbb{Z}/2\mathbb{Z}$	S_3	4	$o-(-4)-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
93	$A_1+D_4+(o-*-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	D_4 or Q_3	4	$*$	$U^0\supseteq\mathbb{C}^2$

94	$A\ +(o-*-o^3-\overset{o}{\overset{\vert}{o}}-o)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	$2A_1+((-4)-o-\overset{o}{\overset{\vert}{o}}-o)$	$V^0\supset\mathbb{C}\times\mathbb{C}^*$
95	$3A_1+(o-*-o-\overset{o}{\overset{\vert}{o}}-o)$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	D_4 or Q_3	3	$o-(-4)-o$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^{**}$
96	$A_1+A_3+(o-*-\overset{o}{\overset{\vert}{o}}-o)$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	4	$4A_1+((-4)-(-4))$	$V^0\supseteq\mathbb{C}\times\mathbb{C}^*$
97	D_7+*	(0)	(0)	1	$\bar{U}=\bar{V}$	$V^0\supset\mathbb{C}\times\mathbb{C}^*$

Let $f: U \to \bar{U}$ be the minimal resolution of singularities on the quasi-universal covering $\bar{U}$ of a dP3 surface $\bar{V}$. The singularity of $\bar{V}$ (resp. $\bar{U}$) is described in terms of the Dynkin graph of $D := g^{-1}(\mathrm{Sing}\ \bar{V}) \subseteq V$ (resp. $B := f^{-1}(\mathrm{Sing}\ \bar{U}) \subseteq U$).

V^0, U^0: stand for $V-D$ and $U-B$, respectively;

$\mathbb{C}^*, \mathbb{C}^{**}, \mathbb{C}^2-P$: stand for $\mathbb{C} - \{0\}$, $\mathbb{C}$ - two distinct points, and $\mathbb{C}^2$ - one point P, respectively;

$\bar{\Sigma}_n (n\geq 2)$: the surface obtained by contracting the minimal section on a Hirzebruch surface Σ_n of degree n;

$*$, o, $(-n)$: stands for the unique (-3) curve on V, a (-2) curve on V, and a $(-n)$ curve on U, respectively;

$*-o^3-*$, etc.: stands for $*—o—o—o—*$, etc., respectively;

$(*-o)+A_m+D_n$: disjoint union of $*-o$ and Dynkin types A_m and D_n;

We employ the following notations for finite groups.

D_4: the dihedral group of order 8;

Q_3: the quaternion group of order 8;

S_3: the third symmetric group of order 6.

In No.22, $\pi_1 = \langle x, y, z \mid x^3=y^3=z^2=1, xy=yx, yz=zy, xz=zx^2\rangle$.

In No.26, $\pi_1 = \langle a, b \mid a^3=b^4=1, ab=ba^2\rangle$.

Though the surfaces $\bar{V}^{(a)}$ and $\bar{V}^{(b)}$ of No.18a and No.18b (resp. No.15a and No.15b) have singularities of the same type, they are not isomorphic to each other.

Table 2

Dynkin type of S	$\pi_1(S^\circ)$	$Cl(S)/Pic(S)$	$\rho(\tilde{S})$	Sing. of $\tilde{S}$
A_1	0	$\mathbb{Z}/2\mathbb{Z}$	1	$\tilde{S} = S$
A_1+A_2	0	$\mathbb{Z}/6\mathbb{Z}$	1	$\tilde{S} = S$
A_4	0	$\mathbb{Z}/5\mathbb{Z}$	1	$\tilde{S} = S$
$2A_1+A_3$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	1	$\tilde{S} = S(A_1)$
D_5	0	$\mathbb{Z}/4\mathbb{Z}$	1	$\tilde{S} = S$
A_1+A_5	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	2	A_2
$3A_2$	$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	1	$\tilde{S} = \mathbb{P}^2$
E_6	0	$\mathbb{Z}/3\mathbb{Z}$	1	$\tilde{S} = S$
$3A_1+D_4$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	1	$\tilde{S} = S(A_1)$
A_7	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	3	A_3
A_1+D_6	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	2	D_4

Dynkin type of S	$\pi_1(S^\circ)$	$C\ell(S)/\mathrm{Pic}(S)$	$\rho(\tilde{S})$	Sing. of $\tilde{S}$
E_7	0	$\mathbb{Z}/2\mathbb{Z}$	1	$\tilde{S} = S$
A_1+2A_3	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$	2	$\tilde{S} = \mathbb{P}^1 \times \mathbb{P}^1$
A_2+A_5	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	3	A_1
D_8	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	3	D_5
$2A_1+D_6$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	3	$\tilde{S} = \widetilde{S(A_7)}$
E_8	0	0	1	$\tilde{S} = S$
A_1+E_7	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	2	E_6
A_1+A_7	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/4\mathbb{Z}$	5	A_1
$2A_4$	$\mathbb{Z}/5\mathbb{Z}$	$\mathbb{Z}/5\mathbb{Z}$	5	a smooth del Pezzo surface of degree 5
A_8	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	5	A_2
$A_1+A_2+A_5$	$\mathbb{Z}/6\mathbb{Z}$	$\mathbb{Z}/6\mathbb{Z}$	4	a smooth del Pezzo surface of degree 6

Dynkin type of S	$\pi_1(S^\circ)$	$C\ell(S)/\mathrm{Pic}(S)$	$\rho(\tilde{S})$	Sing. of $\tilde{S}$
A_2+E_6	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	3	D_4
A_3+D_5	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$	4	A_2
$4A_2$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/3\mathbb{Z})^{\oplus 2}$	1	$\tilde{S} = \mathbb{P}^2$
$2A_1+2A_3$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$	2	$\tilde{S} = \widetilde{S(A_1+2A_3)}$
$2D_4$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	4	$2A_1$

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